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# 学位申請論文

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Smoothing Effects and Dispersion of Singularities

for the Schrödinger Evolution Group

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Abstract

We describe smoothing effects and dispersion of singularities for the Schrödinger evolution group in the weighted Sobolev spaces. Under a fairly general assumption on the potential, it is shown that all singularities in the wavefunction vanish instantly whenever the initial state has sufficient decay. We measure the regularity gained by the wavefunction by the decay property of the initial state. No assumptions on the regularity of the initial state are imposed throughout the paper

## 1. Introduction

In this paper we prove that the Schrödinger evolution group improves regularity (at least locally in space) instantly even when the potential and initial states have local singularities.

Let $H = H_0 + V$ be a Schrödinger operator in $L^2 = L^2(\mathbb{R}^n)$, $n \in \mathbb{N}$, where $H_0 = -(1/2)\Delta$ is the free Hamiltonian and $V$ is a $H_0$-bounded operator of multiplication by a real-valued function, so that $V$ is locally square integrable on $\mathbb{R}^n$ Throughout the paper the $H_0$-bound of $V$ is assumed to be less than one and therefore $H$ is self-adjoint in $L^2$ with domain $D(H) = D(H_0)$ We give a picture of smoothing effects for $e^{-itH}$ mainly in terms of the weighted Sobolev space $H^{m,s}$, $m$, $s \in \mathbb{R}$, defined by

$$H^{m,s} = \{\psi \in \mathcal{S}'; \ \|\psi\|_{m,s} = \|(1+|x|^2)^{s/2}(\mathbb{1}-\Delta)^{m/2}\psi\| < \infty\},$$

where $\|\cdot\|$ denotes the $L^2$-norm. The basic classes of functions in this paper, which turn to be a real vector spaces, are given by

<u>Definition.</u> A real-valued measurable function $W$ on $\mathbb{R}^n$ is said to lie in $\Sigma_1$ if and only if there are constants $0 \leq \lambda < 1$ and $C > 0$ such that

$$\|W\psi\|_{-1,0} \leq C\|\psi\|_{1,0}^{\lambda} \ \|\psi\|^{1-\lambda}, \quad \psi \in H^{1,0}$$

For an integer $k \geq 2$, $W$ is said to lie in $\Sigma_k$ if and only if there are constants $0 \leq \lambda < 1$ and $C > 0$ such that

$$\|W\psi\| \leq C\|\psi\|_{k,0}^{\lambda} \ \|\psi\|^{1-\lambda}, \quad \psi \in H^{k,0}$$

We consider the following assumption on $\hat{V} = V + (1/2)x\cdot\nabla V$

$(A)_k$ (for $k = 1, 2$) $\qquad \hat{V} \in \Sigma_k$

Our goal in this paper is to obtain sufficient conditions for the following $(S)_k$, $k \in \mathbb{N}$:

$(S)_k$

(1) For $t \neq 0$, $e^{-itH}$ is bounded from $H^{0,k}$ to $H^{k,-k}$ and has the estimate

$$\|e^{-itH}\phi\|_{k,-k} \leq C(k)(|t|^{-k} + 1)\|\phi\|_{0,k} \tag{1.1}$$

(2) The map $(\mathbb{R}\setminus\{0\})\times H^{0,k} \ni (t,\phi) \longmapsto e^{-itH}\phi \in H^{k,-k}$ is continuous.

(3) For any $\phi \in H^{0,k}$,

$$\lim_{t\to\pm 0} |t|^k\|e^{-itH}\phi\|_{k,-k} = 0. \tag{1.2}$$

<u>Theorem 1.</u> Let $k = 1$ or $2$. Then, $(A)_k$ implies $(S)_k$

Parts (1)-(2) of $(S)_k$ show the smoothing effects and dispersion of singularities for $e^{-itH}$ In particular, the regularity gained by the wavefunction $e^{-itH}\phi$, $t \neq 0$, can be measured by the decay of $\phi$ These properties have nothing against the time reversibility of $e^{-itH}$, because they require the expence of the weight $(1+|x|^2)^{-k/2}$ of negative exponent. It is therefore reasonable to expect that in spite of the smoothing effects, decay properties may not be preserved under $e^{-itH}$ if $\phi$ has some singularities. This observation is justified by the following example. Let $n = 1$ and let $\phi(x) = -e^{-x}$, $x > 0$; $\phi(x) = e^{x}$, $x < 0$. Then $\phi \in \bigcap_{k\in\mathbb{N}} H^{0,k}$ while $e^{-itH_0}\phi \notin H^{0,1}$ for $t \neq 0$. Moreover, $e^{-itH_0}\phi \in C^{\infty}\cap L^{\infty}$ for $t \neq 0$ (see Theorem 3 below) We have another example $(e^{-itH_0}\delta)(x) = (2\pi it)^{-n/2}\exp(i|x|^2/2t)$, where $\delta$ denotes the Dirac measure, although this extreme example falls out of the scope of the $L^2$-setting. On the other hand, we already know that decay property is preserved if the initial state has sufficient regularity. More precisely, $H^{k,0}\cap H^{0,k}$ is invariant under $e^{-itH}$ for any $t \in \mathbb{R}$ [14][19][21]

We should emphasize that without further assumptions on $V$, $(S)_k$ fails for $k \geq 3$. For example, if $V(x) = -(n-1)/2|x|$, $\phi(x) = e^{-|x|}$, $n \geq 3$, then $\phi \in \bigcap_{k\geq 0} H^{0,k}$ while $e^{-itH}\phi = e^{it/2}\phi \notin H^{n/2+1,0}$ and $(S)_k$ is impossible for $k \geq n/2 + 1$.

For $k \geq 3$, we consider the following assumptions.

$(A)_k$ For all $\alpha \in (\mathbb{N}\cup\{0\})^n$ with $|\alpha| \leq k - 2$, $\partial^{\alpha}\hat{V} \in \Sigma_{2+|\alpha|}$ and $\partial^{\alpha}V\cdot$ is bounded from $H^{2+|\alpha|,0}$ to $L^2$

Theorem 2. Let $k \geq 3$ be an integer Suppose that $D(|H|^{k/2}) = H^{k,0}$ and $(A)_k$ hold. When $k$ is odd, suppose in addition that $(A)_1$ holds. Then $(S)_k$ holds.

Under a more restricted system, every wavefunction becomes smooth and all singularities in the initial state $\phi$ vanish instantly whenever $\phi$ decays rapidly at infinity

Theorem 3. Suppose that $D(|H|^{k/2}) = H^{k,0}$ and $(A)_k$ hold for all $k \in \mathbb{N}$. Then:

(1) For any $\phi \in H^{0,\infty} = \bigcap_{k\geq 0} H^{0,k}$ and any $t \neq 0$, $e^{-itH}\phi \in C^\infty \cap L^\infty$

(2) The map $(\mathbb{R}\setminus\{0\})\times H^{0,\infty} \ni (t,\phi) \longmapsto e^{-itH}\phi \in C^\infty$ is continuous, where $H^{0,\infty}$ is topologized as projective limit.

(3) If in addition, $\partial^\alpha V \in L^\infty$ for all $\alpha \in (\mathbb{N}\cup\{0\})^n$, then for any $\phi \in H^{0,\infty}$, the map $(\mathbb{R}\setminus\{0\})\times\mathbb{R}^n \ni (t,x) \longmapsto (e^{-itH}\phi)(x) \in \mathbb{C}$ is $C^\infty$

In Section 2, Theorems 1-3 will be derived from more general results. We give here some examples of functions $W$ in $\Sigma_k$

Example 1. If $|W|^{1/2}$ is bounded from $H^{\lambda,0}$ to $L^2$ for some $0 \leq \lambda < 1$, then $W \in \Sigma_1$ If $W$ is bounded from $H^{\lambda k,0}$ to $L^2$ for some $0 \leq \lambda < 1$, then $W \in \Sigma_k$. Therefore by an inequality of HERBST [13; Theorem 2.5], $|x|^{-\mu} \in \Sigma_k$ if $0 < \mu < \min(n/2, \max(k,2))$

Example 2. $L^p_{unif} \subset \Sigma_1$ if $p > n/2$ and $p \geq 1$.
$L^p_{unif} \subset \Sigma_k$ if $k \geq 2$, $p > n/k$ and $p \geq 2$. Here

$$L^p_{unif} = \{W \in L^p_{loc}(\mathbb{R}^n);\ \|W\|_{L^p_{unif}} = \sup_{x\in\mathbb{R}^n} \Big(\int_{|x-y|<1} |W(y)|^p\, dy\Big)^{1/p} < \infty\}$$

See Appendix for a proof Note that $L^p + L^\infty \subsetneq L^p_{unif}$.
If $0 < \mu < \min(n/2, \max(k,2))$, then

$$|x|^{-\mu} \in \bigcap_{1\leq p<n/\mu} L^p_{unif} \subset \bigcap_{\max(2,\min(n/2,n/k))<p<n/\mu} L^p_{unif} \subset \Sigma_k.$$

Combining the results given above and in [19; Theorem 5], we have

<u>Theorem 4.</u> Let $V(x) = \sum_{j=1}^{m} \lambda_j |x|^{-\mu_j}$, $\lambda_j \in \mathbb{R}$, $\mu_j > 0$, $m \in \mathbb{N}$, and let $\mu = \max_{1\le j\le m} \mu_j$ Let $k \in \mathbb{N}$. The conditions $(A)_k$ and $D(|H|^{k/2}) = H^{k,0}$ are satisfied in the following cases:

(1) $0 < \mu < \min(2, n/2)$, when $k \le 2$.

(2) $0 < \mu < n/2-1$ $(n = 3, 4)$, $0 < \mu \le 1$ $(n \ge 5)$, when $k = 3$.

(3) $0 < \mu \le 1$ $(n \ge 2k-1)$, when $k \ge 4$.

The most complete description of smoothing effects will be given by the free evolution $U(t) = e^{-itH_0}$ $U(t)$ satisfies the relation $(x + it\nabla)U(t) = U(t)x$, which may be written as

$$\nabla U(t) = (1/it)[U(t),x] \tag{1.3}$$

The estimate (1.1) for the free evolution follows by repeated use of (1.3) The first step in the perturbed problem which we are working will depend on how to regard (1.3)

There is a large literature on the smoothing effects for Schrödinger type equations [2][7][8][9][10][11][12][15][18][24][28] JENSEN [15] proved that if $\partial^\alpha V \in L^\infty$, $|\alpha| \le k$, then for any $\phi \in H^{0,k}$

$$\|e^{-itH}\phi\|_{k,-k} \le C(k)(|t|^{-k} + |t|^k)\|\phi\|_{0,k},$$

and moreover, the associated operator norm satisfies

$$\liminf_{t \to \pm 0} |t|^k \|e^{-itH}\|_{\mathcal{L}(H^{0,k};\, H^{k,-k})} > 0.$$

In [15], the required regularity was derived from the identity

$$\nabla e^{-itH} = (1/it)[e^{-itH}, x] + (\text{remainder}) \tag{1.4}$$

for $k = 1$ and from multiple commutator identities for $k \geq 2$. Even though (1.4) is a natural extension of (1.3), we still stick to (1.3) and write the L.H.S. of (1.4) as

$$\nabla e^{-itH} = (1/it)(U(t)xU(-t) - x)e^{-itH}$$

for $k = 1$ and $\partial^{\alpha} e^{-itH}$ ($|\alpha| = k \geq 2$) as a linear combination of the terms of the form $x^{\beta}U(t)x^{\gamma}U(-t)e^{-itH}$ This requires the analysis of $U(t)x^{\alpha}U(-t)e^{-itH}\phi = (x + it\partial)^{\alpha}e^{-itH}\phi$ and it turns out to be conveninet to view $x + it\nabla$ as one object [7][8][9][10][11][12][18] That operator may be called the generator of Galiei transformations [5] We then reduce the proof of $(S)_k$ to obtaining an *a priori* estimate for $\|U(t)|x|^{k}U(-t)e^{-itH}\phi\|$, which will be derived from an identity involving $\tilde{V} = V + (1/2)x\cdot\nabla V$. This process has been carried out only for the case $k \leq 2$ [6][9][11][12][18]

Throughout the paper we use the following notations. For Banach spaces $X$ and $Y$, $\mathcal{L}(X; Y)$ denotes the Banach space of bounded operators from $X$ to $Y$; for $s \in [0,\infty)$, $[s]$ denotes the largest integer $\leq s$; $[\cdot,\cdot]$ denotes the commutator; $\partial_t = \partial/\partial t$; $\partial_k$ denotes the distributional derivative with respect to the k-th coordinate; $\Delta = \sum_{k=1}^{n} \partial_k^2$ ; $U(t) = e^{-itH_0}$, $t \in \mathbb{R}$; $S(t) = \exp(i|x|^2/2t)\cdot$, $t \in \mathbb{R}\setminus\{0\}$; $x = (x_1,\cdots,x_n)$, $\nabla = \partial = (\partial_1,\cdots,\partial_n)$, $J(t) = (J_1(t),\cdots, J_n(t))$, $J_k(t) = U(t)x_kU(-t)$; $|J|^m(t) = U(t)|x|^mU(-t)$, $m \in [0,\infty)$; for a multi-index $\alpha = (\alpha_1, \quad ,\alpha_n)$ we set $|\alpha| = \sum_{k=1}^{n} \alpha_k$, $\alpha! = \prod_{k=1}^{n} \alpha_k!$, $\binom{\alpha}{\beta} = \alpha!/\beta!(\beta-\alpha)!$ $(\beta \le \alpha)$, $\partial^\alpha = \prod_{k=1}^{n} \partial_k^{\alpha_k}$, $x^\alpha = \prod_{k=1}^{n} x_k^{\alpha_k}$, $J^\alpha = \prod_{k=1}^{n} J_k^{\alpha_k}$, $\partial^0 = x^0 = J^0 = \mathbb{1}$; $A = (1/2i)(x\cdot\nabla + \nabla\cdot x)$; $\mathscr{S}$ denotes the Fréchet space of rapidly decreasing functions from $\mathbb{R}^n$ to $\mathbb{C}$; $\mathscr{S}'$ denotes the dual of $\mathscr{S}$; $L^2$ denotes the Lebesgue space $L^2(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)\otimes\mathbb{C}^n$, with the norm denoted by $\|\cdot\|$; $(\cdot,\cdot)$ denotes the $L^2$-scalar product and various anti-dualities; $C^k(I; E)$ denotes the Fréchet space of k-times continuously differentiable functions from an open interval $I \subset \mathbb{R}$ to a Fréchet space E;

Different constants might be denoted by the same letter C, and if necessary, by $C(*. \quad .*)$ in order to indicate the dependence on the quantities appearing in parentheses. The summation over an empty set is understood to be zero. A function, its value at a point, and the multiplication operator by that function might be denoted by the same symbol when this causes no confusion.

The following relations [10][12] will be frequently uesd in the sequel. $J^\alpha(t) = S(t)(it\partial)^\alpha S(-t)$, $J(t) = S(t)(it\nabla)S(-t)$, $|J|^m(t) = S(t)(-t^2\Delta)^{m/2}S(-t)$, $t \in \mathbb{R}\setminus\{0\}$

## 2. Proof of Theorems 1-3, part 1. Reduction of the problem.

The purpose of this section is to make a reduction of the problem. We consider the following condition $(J)_k$, $k \in \mathbb{N}$:

$$(J)_k \begin{cases} \text{For any } |\alpha| \le k \text{ and any } \phi \in H^{0,k}, \\ \sup_{t\in\mathbb{R}} (1+|t|)^{-|\alpha|} \| J^\alpha(t) e^{-itH}\phi \| \le C\|\phi\|_{0,k} \\ \text{and the map } \mathbb{R} \ni t \longmapsto J^\alpha(t)e^{-itH}\phi \in L^2 \text{ is continuous.} \end{cases}$$

The smoothing effects stated in terms of $J^\alpha$ as in $(J)_k$ seem to be crucial. In fact:

<u>Proposition 2.1.</u> Let $k \in \mathbb{N}$. Then $(J)_k$ implies $(S)_k$.

<u>Proof.</u> For $t \ne 0$, We use the following formulae of operators on $\mathcal{S}'$

$$S(t)(it\partial)^\alpha S(-t) = J^\alpha(t) \tag{2.1}$$

$$S(-t)(it\partial)^\alpha S(t)$$

$$= \sum_{\beta\le\alpha} \sum_{\gamma\le[\beta/2]} \binom{\alpha}{\beta} \frac{\beta!(-1)^{|\beta+\gamma|}}{\gamma!(\beta-2\gamma)!} (it/2)^{|\gamma|} x^{\beta-2\gamma}(it\partial)^{\alpha-\beta}, \tag{2.2}$$

where $[\beta/2] = ([\beta_1/2],\cdots,[\beta_n/2])$ Let $\phi \in H^{0,k}$ By $(J)_k$ and (2.1), $\partial^\alpha S(-t)e^{-itH}\phi = (it)^{-|\alpha|}S(-t)J^\alpha(t)e^{-itH}\phi \in L^2$ for any $|\alpha| \le k$. We let the operators on both sides of (2.2) act on

$S(-t)e^{-itH}\phi \in H^{0,k}$ to obtain

$$(it\partial)^{\alpha}e^{-itH}\phi$$

$$= \sum_{\beta\le\alpha}\ \sum_{\gamma\le[\beta/2]} \binom{\alpha}{\beta}\frac{\beta!(-1)^{|\beta+\gamma|}}{\gamma!(\beta-2\gamma)!}(it/2)^{|\gamma|}x^{\beta-2\gamma}J^{\alpha-\beta}(t)e^{-itH}\phi \qquad (2.3)$$

It follows from $(J)_k$ that every term on the R.H.S. of (2.3) is in $H^{0,-|\alpha|}$ for any $|\alpha| \le k$ and moreover,

$$\|\partial^{\alpha}e^{-itH}\phi\|_{0,-|\alpha|} \le C|t|^{-|\alpha|}\sum_{\beta\le\alpha}(1+|t|^{|\beta|/2})\|J^{\alpha-\beta}(t)e^{-itH}\phi\|$$

$$\le C(|t|^{-|\alpha|}+1)\|\phi\|_{0,k}.$$

This proves part (1) of $(S)_k$, since the norm $|||\psi|||_{k,s} = \|\psi\|_{0,s} + \sum_{|\alpha|=k}\|\partial^{\alpha}\psi\|_{0,s}$ is an equivalent norm on $H^{k,s}$, $s \in \mathbb{R}$ (see TRIEBEL [27; Theorems 1, 3 and 4]) We turn to part (2) Let $(s,\psi) \in (\mathbb{R}\setminus\{0\})\times H^{0,k}$ We have

$$\|e^{-itH}\phi - e^{-isH}\psi\|_{k,-k} \le \|e^{-itH}(\phi - \psi)\|_{k,-k} + \|e^{-itH}\psi - e^{-isH}\psi\|_{k,-k}$$

$$\le C(|t|^{-k}+1)\|\phi - \psi\|_{0,k} + \|e^{-itH}\psi - e^{-isH}\psi\|_{k,-k},$$

which implies the required continuity at $(s,\psi)$, since $(J)_k$ and (2.3) show that $\mathbb{R}\setminus\{0\} \ni t \longmapsto e^{-itH}\psi \in H^{k,-k}$ is continuous. We proceed to part (3) We write the R.H.S. of (2.3) as

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\beta|}x^{\beta}(J^{\alpha-\beta}(t)e^{-itH}\phi - x^{\alpha-\beta}\phi)$$

$$+ \sum_{\beta\le\alpha}\ \sum_{0\ne\gamma\le[\beta/2]}\binom{\alpha}{\beta}\frac{\beta!(-1)^{|\beta+\gamma|}}{\gamma!(\beta-2\gamma)!}(it/2)^{|\gamma|}x^{\beta-2\gamma}J^{\alpha-\beta}(t)e^{-itH}\phi.$$

where we have used $\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(-1)^{|\beta|} = 0$. With the notation as above,

$$\begin{aligned}|t|^{k}|||e^{-itH}\phi|||_{k,-k} &\le |t|^{k}\|e^{-itH}\phi\|_{0,-k} + \sum_{|\alpha|=k}\|(it\partial)^{\alpha}e^{-itH}\phi\|_{0,-k} \\ &\le |t|^{k}\|\phi\| + \sum_{|\alpha|=k}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\|J^{\alpha-\beta}(t)e^{-itH}\phi - x^{\alpha-\beta}\phi\| \\ &\quad + C\sum_{|\alpha|=k}\sum_{\beta\le\alpha}\sum_{0\ne\gamma\le[\beta/2]}|t|^{|\gamma|}\|J^{\alpha-\beta}(t)e^{-itH}\phi\|\end{aligned}$$

$(J)_k$ shows that the R.H.S. of the last inequality converges to zero as $t \longrightarrow \pm 0$. Q.E.D.

<u>Proposition 2.2.</u> Suppose that $(J)_k$ holds for any $k \in \mathbb{N}$. Then:

(1) For any $\phi \in H^{0,\infty}$ and any $t \in \mathbb{R}\setminus\{0\}$, $e^{-itH}\phi \in C^{\infty}\cap L^{\infty}$

(2) The map $(\mathbb{R}\setminus\{0\})\times H^{0,\infty} \ni (t,\phi) \longmapsto e^{-itH}\phi \in C^{\infty}$ is continuous.

<u>Proof.</u> Let $\phi \in H^{0,\infty}$ and $t \ne 0$. In the same way as in the proof of Proposition 2.1, $S(-t)e^{-itH}\phi \in \bigcap_{k\in\mathbb{N}} H^{k,0}$ By Sobolev's lemma, $S(-t)e^{-itH}\phi \in C^{\infty}\cap L^{\infty}$ In particular, $e^{-itH}\phi \in L^{\infty}$ By applying (2.2) to $S(-t)e^{-itH}\phi$, we see that $e^{-itH}\phi \in C^{\infty}$

We turn to part (2) We prove the required continuity at $(s,\psi) \in (\mathbb{R}\setminus\{0\})\times H^{0,\infty}$ Let $m \in \mathbb{N}$, $R > 0$, $N = [n/2]+1$.
For $f \in C^{\infty}$, we set

$$|f|_{m,R} = \sum_{|\alpha|\le m}\sup_{|x|\le R}|\partial^{\alpha}f(x)|$$

We have

$$|e^{-itH}\phi - e^{-isH}\psi|_{m,R} \le |e^{-itH}\psi - e^{-isH}\psi|_{m,R} + |e^{-itH}(\phi - \psi)|_{m,R}. \quad (2.4)$$

In order to estimate the first semi-norm on the R.H.S. of (2.4), by using (2.2), we write

$$\partial^{\alpha}(e^{-itH_{\psi}} - e^{-isH_{\psi}})$$

$$= \sum_{\beta\le\alpha} \sum_{\gamma\le[\beta/2]} 2^{-|\gamma|}\binom{\alpha}{\beta}\frac{\beta!(-i)^{|\beta-\gamma|}}{\gamma!(\beta-2\gamma)!}x^{\beta-2\gamma}\Big(S(t)t^{-|\beta-\gamma|}F_{\alpha\beta\gamma}(t,s)$$

$$+ (t^{-|\beta-\gamma|}S(t) - s^{-|\beta-\gamma|}S(s))\partial^{\alpha-\beta}S(-s)e^{-isH_{\psi}}\Big),$$

where

$$F_{\alpha\beta\gamma}(t,s) = \partial^{\alpha-\beta}(S(-t)e^{-itH_{\psi}} - S(-s)e^{-isH_{\psi}})$$

By Sobolev's lemma and the inequality

$$|t^{-j}\exp(|x|^2/2it) - s^{-j}\exp(|x|^2/2is)|$$

$$\le |t^{-j} - s^{-j}| + 2^{-1}|x|^2 s^{-j}|t^{-1} - s^{-1}|,$$

the first semi-norm on the R.H.S. of (2.4) is estimated by

$$C(1+R)^m \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \sum_{\gamma\le[\beta/2]} |t|^{-|\beta-\gamma|} \sum_{|\tilde{\alpha}|\le N} \|\partial^{\tilde{\alpha}}F_{\alpha\beta\gamma}(t,s)\|$$

$$+ C(1+R)^{m+2} \sum_{j=1}^{m} |t^{-j}-s^{-j}| \sum_{|\tilde{\alpha}|\le N+m} \|\partial^{\tilde{\alpha}}S(-s)e^{-isH_{\psi}}\|$$

Setting $\tilde{\beta} = \tilde{\alpha} + \alpha - \beta$, we have by (2.1)

$$\|\partial^{\tilde{\alpha}}F_{\alpha\beta\gamma}(t,s)\|$$

$$= \|S(-t)((it)^{-1}J(t))^{\tilde{\beta}}e^{-itH_{\psi}} - S(-s)((is)^{-1}J(s))^{\tilde{\beta}}e^{-isH_{\psi}}\|$$

$$\le |t|^{-|\tilde{\beta}|}\|J^{\tilde{\beta}}(t)e^{-itH}\psi - J^{\tilde{\beta}}(s)e^{-isH}\psi\| + |t^{-|\tilde{\beta}|}-s^{-|\tilde{\beta}|}|\|J^{\tilde{\beta}}(s)e^{-isH}\psi\|$$

$$+ |s|^{-|\tilde{\beta}|}\|(S(-t) - S(-s))J^{\tilde{\beta}}(s)e^{-isH}\psi\|,$$

$$\|\partial^{\tilde{\alpha}}S(-s)e^{-isH}\psi\| = |s|^{-|\tilde{\alpha}|}\|J^{\tilde{\alpha}}(s)e^{-isH}\psi\|$$

Therefore, we obtain from $(J)_k$

$$|e^{-itH}\psi - e^{-isH}\psi|_{m,R}$$

$$\le C(1+R)^m(1+|t|^{-m-N})\sum_{|\alpha|\le m+N}\|J^{\alpha}(t)e^{-itH}\psi - J^{\alpha}(s)e^{-isH}\psi\|$$

$$+ C(1+R)^m(1+|t|^{-m})(1+|s|^{m+N})\|\psi\|_{0,m+N}\sum_{j=1}^{m+N}|t^{-j}-s^{-j}|$$

$$+ C(1+R)^m(1+|t|^{-m})(1+|s|^{-m-N})\sum_{|\alpha|\le m+N}\|(S(-t) - S(-s))J^{\alpha}(s)e^{-isH}\psi\|$$

$$+ C(1+R)^{m+2}(1+|s|^{-m-N})\|\psi\|_{0,m+N}\sum_{j=1}^{m}|t^{-j}-s^{-j}| \equiv \mathrm{I} \tag{2.5}$$

A similar and simpler calculation shows

$$|e^{-itH}(\phi - \psi)|_{m,R} \le C(1+R)^m(1+|t|^{-m-N})\|\phi - \psi\|_{0,m+N} \equiv \mathrm{II},$$

and hence $|e^{-itH}\phi - e^{-isH}\psi|_{m,R} \le \mathrm{I} + \mathrm{II}$, which proves the required continuity Q.E.D.

## 3. Proof of Theorems 1-3, part 2.

It remains to prove $(J)_k$ We collect here some preliminary results. For $k \in \mathbb{N}\cup\{0\}$, we set $\mathcal{H}_k = H^{k,0} \cap H^{0,k}$ $\mathcal{H}_k$ is a Hilbert space with the norm $|||\psi|||_k = (\|\psi\|^2_{k,0} + \|\psi\|^2_{0,k})^{1/2}$

Lemma 3.1 ([19; Lemma 2.2][16]) For any $k \in \mathbb{N}$, $\mathcal{H}_k \subset \bigcap_{j=0}^{k} H^{j,k-j}$ and for any $0 \le j \le k$,

$$\sum_{|\alpha|=j} \|\partial^\alpha \psi\|_{0,k-j} \le C(k,j) \sum_{|\beta|=k} \|\partial^\beta \psi\|^{j/k} \|\psi\|^{1-j/k}_{0,k}, \quad \psi \in \mathcal{H}_k$$

Lemma 3.2. For any $j, \ell \in \mathbb{N}\cup\{0\}$, $\alpha \in (\mathbb{N}\cup\{0\})^n$ and any $t \in \mathbb{R}$, $J^\alpha(t) \in \mathcal{L}(\mathcal{H}_{|\alpha|+j+\ell}; H^{j,\ell})$ and the map $\mathbb{R} \ni t \longmapsto J^\alpha(t) \in \mathcal{L}(\mathcal{H}_{|\alpha|+j+\ell}; H^{j,\ell})$ is continuous. Moreover, for any $j \in \mathbb{N}\cup\{0\}$,

$$|||J^\alpha(t)\psi|||_j \le C(1+|t|)^{j+|a|} |||\psi|||_{j+|\alpha|}, \quad t \in \mathbb{R}, \quad \psi \in \mathcal{H}_{j+|\alpha|}$$

Proof. By Lemma 3.1, $x^\beta \partial^\gamma \in \mathcal{L}(\mathcal{H}_{j+\ell+|\beta+\gamma|}; H^{j,\ell})$ for all $\beta$, $\gamma$. Therefore the lemma follows from the formula

$$J^\alpha(t) = \sum_{\beta\le\alpha} \sum_{\gamma\le[\beta/2]} \binom{\alpha}{\beta} \frac{\beta!\,2^{-|\gamma|}}{\gamma!\,(\beta-2\gamma)!} (it)^{|\gamma|+|\alpha-\beta|} x^{\beta-2\gamma} \partial^{\alpha-\beta} \tag{3.1}$$

Q.E.D.

Lemma 3.3 ([19; Theorem 1][21][14]) Let $k \in \mathbb{N}$. Suppose that $D(|H|^{k/2}) = H^{k,0}$ when $k \ge 3$. Then:

(1) $\mathcal{H}_k$ and $H^{k,0}$ are invariant under $e^{-itH}$ for any $t \in \mathbb{R}$.

(2) The map $(t,\phi) \longmapsto e^{-itH}\phi$ is continuous from $\mathbb{R}\times\mathcal{H}_k$ to $\mathcal{H}_k$ and from $\mathbb{R}\times H^{k,0}$ to $H^{k,0}$ Moreover,

$$|||e^{-itH}\phi|||_k \le C(k)(1+|t|)^k|||\phi|||_k, \quad t \in \mathbb{R}, \quad \phi \in \mathcal{H}_k,$$

$$\|e^{-itH}\phi\|_{k,0} \le C(k)\|\phi\|_{k,0}, \quad t \in \mathbb{R}, \quad \phi \in H^{k,0}$$

(3) For any $|\alpha| \le k$ and $\phi \in \mathcal{H}_k$, the map $\mathbb{R} \ni t \longmapsto e^{itH}x^\alpha e^{-itH}\phi \in L^2$ is continuously differentiable and

$$\frac{d}{dt}(e^{itH}x^\alpha e^{-itH}\phi) = -ie^{itH}((1/2)(\Delta x^\alpha) + (\nabla x^\alpha)\cdot\nabla)e^{-itH}\phi.$$

Lemma 3.4. Let $k \in \mathbb{N}$ and suppose that $D(|H|^{k/2}) = H^{k,0}$ when $k \ge 3$. Let $\phi \in \mathcal{H}_k$. Then:

(1) For any $|\alpha| \le k$ and $t \in \mathbb{R}$, $J^\alpha(t)e^{-itH}\phi \in \mathcal{H}_{k-|\alpha|}$ and the map $\mathbb{R} \ni t \longmapsto J^\alpha(t)e^{-itH}\phi \in \mathcal{H}_{k-|\alpha|}$ is continuous. Moreover, for any $j,\ \ell \in \mathbb{N}\cup\{0\}$ with $j + \ell \le k - |\alpha|$,

$$\|J^\alpha(t)e^{-itH}\phi\|_{j,\ell} \le C(1+|t|)^{|\alpha|+\ell}|||\phi|||_{j+\ell+|\alpha|}$$

(2) The map $\mathbb{R} \ni t \longmapsto |J|^k(t)e^{-itH}\phi \in L^2$ is continuous. Moreover,

$$\|\,|J|^k(t)e^{-itH}\phi\| \le C(1+|t|)^k|||\phi|||_k.$$

(3) For any $t \in \mathbb{R}\setminus\{0\}$, $S(-t)e^{-itH}\phi \in \mathcal{H}_k$ and the map $\mathbb{R}\setminus\{0\} \ni t \longmapsto S(-t)e^{-itH}\phi \in \mathcal{H}_k$ is continuous.

Proof. By Lemma 3.2, $\mathbb{R} \ni t \longmapsto J^\alpha(t) \in \mathcal{L}(\mathcal{H}_k;\ \mathcal{H}_{k-|\alpha|})$ is continuous. By Lemma 3.3, $\mathbb{R} \ni t \longmapsto J^\alpha(t)e^{-itH}\phi \in \mathcal{H}_{k-|\alpha|}$ is continuous. By Lemmas 3.1 and 3.3,

$$\|x^{\beta}\partial^{\gamma}e^{-itH}\phi\| \le C(1+|t|)^{|\beta|}|||\phi|||_{|\beta+\gamma|}$$

Therefore by (3.1),

$$\|J^{\alpha}(t)e^{-itH}\phi\|_{j,\ell} \le C(1+|t|)^{|\alpha|+\ell}|||\phi|||_{j+\ell+|\alpha|}$$

This proves part (1) We turn to part (2) By part (1),

$$\||J|^k(t)e^{-itH}\phi\|^2 = \||x|^k U(-t)e^{-itH}\phi\|^2$$

$$= \sum_{|\alpha|=k} \frac{k!}{\alpha!}\|J^{\alpha}(t)e^{-itH}\phi\|^2 \le C(1+|t|)^{2k}|||\phi|||_k^2$$

The required continuity at $s \in \mathbb{R}$ follows from the estimate

$$\||J|^k(t)e^{-itH}\phi - |J|^k(s)e^{-isH}\phi\|^2$$

$$\le 2\||x|^k(U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi)\|^2 + 2\|(U(t)-U(s))|x|^k U(-s)e^{-isH}\phi\|^2$$

$$\le 4\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|J^{\alpha}(t)e^{-itH}\phi - J^{\alpha}(s)e^{-isH}\phi\|^2$$

$$+ 4\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|(U(s)-U(t))x^{\alpha}U(-s)e^{-isH}\phi\|^2$$

$$+ 2\|(U(t)-U(s))|x|^k U(-s)e^{-isH}\phi\|^2$$

This proves part (2) Part (3) follows from the formula

$$\partial^{\alpha}S(-t)e^{-itH}\phi$$

$$= \sum_{\beta\le\alpha}\sum_{\gamma\le[\beta/2]}\binom{\alpha}{\beta}\frac{\beta!\,2^{-|\gamma|}}{\gamma!(\beta-2\gamma)!}(it)^{-|\beta-\gamma|}S(-t)x^{\beta-2\gamma}\partial^{\alpha-\beta}e^{-itH}\phi. \qquad \text{Q.E.D.}$$

<u>Lemma 3.5.</u> Let $k \in \mathbb{N}$ and suppose that $D(|H|^{k/2}) = H^{k,0}$ when $k \ge 3$.

(1) If $k = 2m$, $m \in \mathbb{N}$, then there is a constant $C > 0$ such that

$$C^{-1}(\|(-\Delta)^m\psi\|^2 + \|\psi\|^2) \le \|H^m\psi\|^2 + \|\psi\|^2 \le C(\|(-\Delta)^m\psi\|^2 + \|\psi\|^2), \quad \psi \in H^{k,0}$$

(2) If $k = 2m+1$, $m \in \mathbb{N}\cup\{0\}$, then there is a constant $C > 0$ such that

$$C^{-1}(\|(-\Delta)^{k/2}\psi\|^2 + \|\psi\|^2) \le (1/2)\|\nabla H^m\psi\|^2 + (VH^m\psi, H^m\psi) + \|\psi\|^2$$
$$\le C(\|(-\Delta)^{k/2}\psi\|^2 + \|\psi\|^2), \quad \psi \in H^{k,0}$$

Proof. Part (1) follows from the closed graph theorem. We prove part (2) There is a constant $C > 0$ such that

$$C^{-1}(\|\nabla H^m\psi\|^2 + \|\psi\|^2) \le (1/2)\|\nabla H^m\psi\|^2 + (VH^m\psi, H^m\psi) + \|\psi\|^2$$
$$\le C(\|\nabla H^m\psi\|^2 + \|\psi\|^2), \tag{3.2}$$

since the $H_0$-form-bound of $V$ is less than one. By the Heinz-Kato theorem [26], $D(|H|^{\ell/2}) = H^{\ell,0}$ for any $1 \le \ell \le k$. Therefore

$$\|\nabla H^m\psi\|^2 \le C\||H|^{1/2}H^m\psi\|^2 + C\|H^m\psi\|^2$$
$$\le C\||H|^{k/2}\psi\|^2 + C\||H|^{k/2}\psi\|^{4m/k}\|\psi\|^{1-4m/k}$$
$$\le C\||H|^{k/2}\psi\|^2 + C\|\psi\|^2 \le C\|(-\Delta)^{k/2}\psi\|^2 + C\|\psi\|^2,$$

where we have used the moment inequality [26] In view of (3.2), it remains to prove that $\|(-\Delta)^{k/2}\psi\| \le C\|\nabla H^m\psi\| + C\|\psi\|$ In the same way as in the preceding argument, we obtain

$$\|(-\Delta)^{k/2}\psi\| \le C\||H|^{1/2}H^m\psi\| + C\|\psi\| \le C\|\nabla H^m\psi\| + C\|H^m\psi\| + C\|\psi\|$$

$$\leq C\|\nabla H^m\psi\| + C\|(-\Delta)^m\psi\| + C\|\psi\|,$$

which implies the desired estimate since $m < k/2$. Q.E.D.

We introduce the following operators.

$$K(t) = |J|^2(t) + 2t^2V, \quad \mathcal{K}(t) = e^{itH}K(t)e^{-itH}, \quad \hat{\mathcal{V}}(t) = e^{itH}\tilde{V}e^{-itH}$$

We set $K^j(t) = (K(t))^j$, $j \in \mathbb{N}$. Note that

$$K(t) = |x|^2 - 2tA + 2t^2H, \quad t \in \mathbb{R},$$

$$K(t) = 2t^2S(t)HS(-t), \quad t \in \mathbb{R}\setminus\{0\}$$

From now on unless otherwise specified, the special number denoted by $k$ represents the associated assumption in Theorems 1-2. In what follows we often treat the cases $k$ = even and $k$ = odd, separately In the case $k$ = odd, we use a duality argument. For instance, we make extensive use of the facts that $H, V, \tilde{V} \in \mathcal{L}(H^{1,0}; H^{-1,0})$, $|x|^2, A \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_1')$, where $\mathcal{H}_1'$ denotes the strong dual of $\mathcal{H}_1$ $\mathcal{H}_1'$ is identified with the Banach space $H^{-1,0} + H^{0,-1}$

Lemma 3.6. Let $k \in \mathbb{N}$. Let $j, \ell_1, \ell_2 \in \mathbb{N}\cup\{0\}$ satisfy $\ell_1 + \ell_2 + 2j \leq k$. Then for any $t \in \mathbb{R}$, $K^j(t) \in \mathcal{L}(\mathcal{H}_{2j+\ell_1+\ell_2}; H^{\ell_1,\ell_2})$ and the map $\mathbb{R} \ni t \longmapsto K^j(t) \in \mathcal{L}(\mathcal{H}_{2j+\ell_1+\ell_2}; H^{\ell_1,\ell_2})$ is continuous. Moreover, for any $\ell \leq k - 2j$,

$$|||K^j(t)\psi|||_\ell \leq C(1+|t|)^{\ell+2j}|||\psi|||_{\ell+2j}, \quad t \in \mathbb{R}, \ \psi \in \mathcal{H}_{\ell+2j}$$

Proof. By Lemma 3.2, $\mathbb{R} \ni t \longmapsto |J|^{2j}(t) \in \mathcal{L}(\mathcal{H}_{2j+\ell_1+\ell_2}; H^{\ell_1,\ell_2})$ and $\mathbb{R} \ni t \longmapsto J^{\beta}(t) \in \mathcal{L}(\mathcal{H}_{2j+\ell_1+\ell_2}; H^{2j+\ell_1-|\beta|,\ell_2})$ are continuous for all $|\beta| \le 2j$ If $|\alpha_1 + \quad + \alpha_h + \beta| = 2j - 2h$, $1 \le h \le j$, then $\prod_{p=1}^{h} \partial^{\alpha_p} V \in \mathcal{L}(H^{2j+\ell_1-|\beta|,\ell_2}; H^{\ell_1,\ell_2})$ Indeed, for $\ell_2 = 0$ this follows by a direct calculation. For $\ell_2 \ge 1$ we only have to notice that $\|\psi\|^*_{m,s} = \|(\mathbb{1}-\Delta)^{m/2}(1+|x|^2)^{s/2}\psi\|$ is an equivalent norm on $H^{m,s}$ for any $m, s \in \mathbb{R}$. The lemma now follows from the identity

$$K^j(t) = |J|^{2j}(t) + \sum_{h=1}^{j} \sum_{\substack{|\beta|\le 2j-2h \\ |\alpha_1+\cdots+\alpha_h+\beta|=2j-2h}} C(j,h,\{\alpha_p\},\beta)t^{2j-|\beta|}\left(\prod_{p=1}^{h} \partial^{\alpha_p} V\right) J^{\beta}(t) \tag{3.3}$$

Q.E.D.

Lemma 3.7. Let $k \in \mathbb{N}$ and let $\phi \in \mathcal{H}_k$ Let $j, \ell \in \mathbb{N}\cup\{0\}$ satisfy $2j + \ell \le k$. Then for any $t \in \mathbb{R}$, $K^j(t)e^{-itH}\phi \in \mathcal{H}_\ell$ and the map $\mathbb{R} \ni t \longmapsto K^j(t)e^{-itH}\phi \in \mathcal{H}_\ell$ is continuous. Moreover, for any $j, \ell_1, \ell_2 \in \mathbb{N}\cup\{0\}$ with $2j + \ell_1 + \ell_2 \le k$,

$$\|K^j(t)e^{-itH}\phi\|_{\ell_1,\ell_2} \le C(1+|t|)^{2j+\ell_2} |||\phi|||_{2j+\ell_1+\ell_2}, \quad t \in \mathbb{R}.$$

Proof. By Lemma 3.6, $\mathbb{R} \ni t \longmapsto K^j(t) \in \mathcal{L}(\mathcal{H}_{2j+\ell}; \mathcal{H}_\ell)$ is continuous. By Lemma 3.3, $\mathbb{R} \ni t \longmapsto K^j(t)e^{-itH}\phi \in \mathcal{H}_\ell$ is continuous. The inequality in the lemma follows from (3.3) and Lemma 3.4. Q.E.D.

Lemma 3.8. Let $k \in \mathbb{N}$ and let $\phi \in \mathcal{H}_k$

(1) If $k = 2m$, $m \in \mathbb{N}$, then the map $\mathbb{R} \ni t \longmapsto \mathcal{X}^m(t)\phi \in L^2$ is continuously differentiable and

$$\frac{d}{dt}\mathcal{X}^m(t)\phi = 4t\sum_{j=1}^{m}\mathcal{X}^{j-1}(t)\hat{\mathcal{V}}(t)\mathcal{X}^{m-j}(t)\phi.$$

(2) If $k = 2m+1$, $m \in \mathbb{N}\cup\{0\}$, then the map $\mathbb{R} \ni t \longmapsto \mathcal{X}^m(t)\phi \in \mathcal{H}_1$ is continuously differentiable and the identity in part (1) holds with every term in $\mathcal{H}_1$

<u>Proof.</u> (1) We first consider the case $m = 1$. Let $\phi \in \mathcal{H}_2$. Then

$$\mathcal{X}(t)\phi = e^{itH}|x|^2e^{-itH}\phi - 2te^{itH}Ae^{-itH}\phi + 2t^2H\phi.$$

By Lemma 3.3,

$$\frac{d}{dt}(e^{itH}|x|^2e^{-itH}\phi) = 2e^{itH}Ae^{-itH}\phi.$$

In the same way as in the proof of the Virial theorem [3][17][20], we regularize $A$ by $Ai\lambda R_\lambda = Ai\lambda(A+i\lambda)^{-1}$, $\lambda \in \mathbb{R}\setminus\{0\}$, and integrate the derivative

$$\begin{aligned}\frac{d}{dt}(e^{itH}Ai\lambda R_\lambda e^{-itH}\phi) &= ie^{itH}i\lambda R_\lambda[H,A]i\lambda R_\lambda e^{-itH}\phi \\ &= 2e^{itH}i\lambda R_\lambda(H - \tilde{V})i\lambda R_\lambda e^{-itH}\phi\end{aligned}$$

We then take the limit $\lambda \longrightarrow \infty$ in the resulting identity by using Lebesgue's dominated convergence theorem, Lemma 3.3 and [3; Lemma 4.5] to obtain

$$e^{itH}Ae^{-itH}\phi = A\phi + 2\int_0^t e^{isH}(H - \tilde{V})e^{-isH}\phi\, ds$$

$$= A\phi + 2tH\phi - 2\int_0^t e^{isH}\tilde{V}e^{-isH}\phi\, ds.$$

This proves that $\mathbb{R} \ni t \longmapsto e^{itH}Ae^{-itH}\phi \in L^2$ is continuously differentiable and

$$\frac{d}{dt}(e^{itH}Ae^{-itH}\phi) = 2H\phi - 2e^{itH}\tilde{V}e^{-itH}\phi.$$

Therefore, $\mathbb{R} \ni t \longmapsto \mathcal{X}(t)\phi \in L^2$ is continuously differentiable and

$$\frac{d}{dt}\mathcal{X}(t)\phi = 4te^{itH}\tilde{V}e^{-itH}\phi = 4t\tilde{\mathcal{V}}(t)\phi. \tag{3.4}$$

We proceed to the case $m \geq 2$. We first remark that $\mathbb{R} \ni t \longmapsto \mathcal{X}^{j-1}(t)\tilde{\mathcal{V}}(t)\mathcal{X}^{m-j}(t)\phi \in L^2$ is continuous for any $1 \leq j \leq m$. Indeed, this follows from the continuity of $\mathbb{R} \ni t \longmapsto K^{j-1}(t)\tilde{V}K^{m-j}(t)e^{-itH}\phi \in L^2$ and the inequality

$$\|K^{j-1}(t)\tilde{V}K^{m-j}(t)e^{-itH}\phi\| \leq C(1+|t|)^{2m-2}|||\phi|||_{2m},$$

both of which are derived from Lemmas 3.3 and 3.7, where we should note that $\tilde{V} \in \mathcal{L}(\mathcal{H}_{2j};\ \mathcal{H}_{2j-2})$ Now we write for $\psi \in \mathcal{H}_{2m}$ and $h \neq 0$

$$(h^{-1}(\mathcal{X}^m(t+h)-\mathcal{X}^m(t))\phi - 4t\sum_{j=1}^{m}\mathcal{X}^{j-1}(t)\tilde{\mathcal{V}}(t)\mathcal{X}^{m-j}(t)\phi, \psi)$$

$$= \sum_{j=1}^{m}((h^{-1}(\mathcal{X}(t+h)-\mathcal{X}(t)) - 4t\tilde{\mathcal{V}}(t))\mathcal{X}^{m-j}(t)\phi,\ \mathcal{X}^{j-1}(t+h)\psi)$$

$$+ 4t\sum_{j=2}^{m}\sum_{\ell=1}^{j-1}((\mathcal{X}(t+h)-\mathcal{X}(t))\mathcal{X}^{j-\ell-1}(t)\tilde{\mathcal{V}}(t)\mathcal{X}^{m-j}(t)\phi, \mathcal{X}^{\ell-1}(t+h)\psi) \tag{3.5}$$

By Lemma 3.7 and (3.4), the R.H.S. of (3.5) converges to zero as

$h \longrightarrow 0$. We have thus proved

$$(\mathcal{X}^m(t)\phi,\psi) = (|x|^{2m}\phi,\psi) + 4\sum_{j=1}^{m}\int_0^t s(\mathcal{X}^{j-1}(s)\hat{\gamma}(s)\mathcal{X}^{m-j}(s)\phi,\psi)ds. \qquad (3.6)$$

We already know that for any $1 \le j \le m$,
$\mathbb{R} \ni t \longmapsto t\mathcal{X}^{j-1}(t)\hat{\gamma}(t)\mathcal{X}^{m-j}(t)\phi \in L^2$ is continuous and

$$\|t\mathcal{X}^{j-1}(t)\hat{\gamma}(t)\mathcal{X}^{m-j}(t)\phi\| \le C(1+|t|)^{2m-1}|||\phi|||_{2m}.$$

Therefore (3.6) still holds for any $\psi \in L^2$ and
$\mathbb{R} \ni t \longmapsto t\mathcal{X}^{j-1}(t)\hat{\gamma}(t)\mathcal{X}^{m-j}(t)\phi \in L^2$ is integrable. Consequently, the integral and the continuous linear functional $(\cdot,\psi)$ on $L^2$ commute, so that

$$\mathcal{X}^m(t)\phi = |x|^{2m}\phi + 4\sum_{j=1}^{m}\int_0^t s\mathcal{X}^{j-1}(s)\hat{\gamma}(s)\mathcal{X}^{m-j}(s)\phi\, ds \qquad (3.7)$$

holds in $L^2$ This proves part (1)
(2) By Lemmas 3.3 and 3.7, $\mathbb{R} \ni t \longmapsto \mathcal{X}^m(t)\phi \in \mathcal{H}_1$ and $\mathbb{R} \ni t \longmapsto \mathcal{X}^{j-1}(t)\hat{\gamma}(t)\mathcal{X}^{m-j}(t)\phi \in \mathcal{H}_1$ are continuous. Since the integrals in (3.7) converge in $\mathcal{H}_1$, as in the derivation of (3.7) from (3.6), we see that the identity (3.7) holds in $\mathcal{H}_1$ Q.E.D.

<u>Lemma 3.9.</u> Let $k = 2m+1$, $m \in \mathbb{N}\cup\{0\}$ Then:
(1) For any $j \in \mathbb{N}$ with $j \le m+1$ and any $t \in \mathbb{R}$, $K^j(t) \in \mathcal{L}(\mathcal{H}_{2j-1};\, \mathcal{H}_1')$ and the map $\mathbb{R} \ni t \longmapsto K^j(t) \in \mathcal{L}(\mathcal{H}_{2j-1};\, \mathcal{H}_1')$ is continuous. Moreover,

$$\|K^j(t)\psi\|_{\mathcal{H}_1'} \le C(1+|t|)^{2j}|||\psi|||_{2j-1},\ t \in \mathbb{R},\ \psi \in \mathcal{H}_{2j-1}$$

(2) For any $j \in \mathbb{N}$ with $j \leq m+1$, any $t \in \mathbb{R}$ and any $\phi \in \mathcal{H}_k$, $K^j(t)e^{-itH}\phi \in \mathcal{H}_1'$ and the map $\mathbb{R} \ni t \longmapsto K^j(t)e^{-itH}\phi \in \mathcal{H}_1'$ is continuous. Moreover,

$$\|K^j(t)e^{-itH}\phi\|_{\mathcal{H}_1'} \leq C(1+|t|)^{2j}|||\psi|||_{2j-1}, \quad t \in \mathbb{R}, \ \phi \in \mathcal{H}_{2j-1}$$

(3) For any $\phi \in \mathcal{H}_k$, the map $\mathbb{R} \ni t \longmapsto \mathcal{X}^{m+1}(t)\phi \in \mathcal{H}_1'$ is continuously differentiable and

$$\frac{d}{dt}\,\mathcal{X}^{m+1}(t)\phi = 4t\sum_{j=1}^{m+1} \mathcal{X}^{j-1}(t)\tilde{\gamma}(t)\mathcal{X}^{m+1-j}(t)\phi$$

<u>Proof.</u> We first remark that $\|K(t)\|_{\mathcal{L}(\mathcal{H}_1;\ \mathcal{H}_1')} \leq C(1+|t|)^2$ and that $\mathbb{R} \ni t \longmapsto K(t) \in \mathcal{L}(\mathcal{H}_1;\ \mathcal{H}_1')$ is continuous. Therefore parts (1)-(2) follow from Lemmas 3.3 and 3.7 We turn to part (3) This is an analogue of Lemma 3.8 and can be proved in an analogous way by working in $\mathcal{H}_1'$ When $m = 0$, we write for $\phi,\ \psi \in \mathcal{H}_1$

$$(\mathcal{X}(t)\phi,\psi)$$

$$= (e^{itH}xe^{-itH}\phi, e^{itH}xe^{-itH}\psi) - 2t(Ae^{-itH}\phi, e^{-itH}\psi) + 2t^2(H\phi,\psi)$$

In a way similar to the proof of Lemma 3.8, we see that $\mathbb{R} \ni t \longmapsto (\mathcal{X}(t)\phi,\psi) \in \mathbb{C}$ is continuously differentiable and

$$\frac{d}{dt}(\mathcal{X}(t)\phi,\psi) = 4t(\tilde{\gamma}(t)\phi,\psi)$$

By Lemma 3.3, $\mathbb{R} \ni t \longmapsto \tilde{\gamma}(t)\phi \in \mathcal{H}_1'$ is continuous and therefore as in the derivation of (3.7) from (3.6), we conclude that part (3) holds for $m = 0$ Since we already know part (2), the proof for the case

$m \geq 1$ is parallel to that of Lemma 3.8. Q.E.D.

Lemma 3.10. (1) Let $k = 2m$, $m \in \mathbb{N}$. Then there are constants $0 \leq \lambda < 1$ and $C > 0$ such that for any $\psi \in H^{k,0}$

$$\sum_{j=1}^{m} \|H^{j-1}\tilde{V}H^{m-j}\psi\| \leq C(\|H^m\psi\| + \|\psi\|)^{\lambda}\|\psi\|^{1-\lambda} \tag{3.8}$$

(2) Let $k = 2m+1$, $m \in \mathbb{N}\cup\{0\}$ Then there are constants $0 \leq \lambda < 1$ and $C > 0$ such that for any $\psi \in H^{k,0}$

$$\sum_{j=1}^{m} \|H^{j}\tilde{V}H^{m-j}\psi\|_{-1,0} \leq C(\|\nabla H^m\psi\| + \|\psi\|)^{\lambda}\|\psi\|^{1-\lambda}$$

Proof. (1) It suffices to prove that the j-th term on the L.H.S. of (3.8) is estimated by the term of the form

$$C(j)(\|H^m\psi\|+\|\psi\|)^{\lambda_j}\|\psi\|^{1-\lambda_j},$$

$0 \leq \lambda_j < 1$. It follows from the Heinz-Kato theorem [26] that $D(|H|^{\ell}) = H^{2\ell,0}$, $1 \leq \ell \leq m$. Therefore

$$\|H^{j-1}\tilde{V}H^{m-j}\psi\| \leq C\|\tilde{V}H^{m-j}\psi\|_{2j-2,0} \leq C\sum_{|\alpha+\beta|\leq 2j-2}\|\partial^{\alpha}\tilde{V}\cdot\partial^{\beta}H^{m-j}\psi\|$$

By assumption, there is a constant $0 \leq \lambda(\alpha) < 1$ such that

$$\begin{aligned}\|\partial^{\alpha}\tilde{V}\cdot\partial^{\beta}H^{m-j}\psi\| &\leq C\|\partial^{\beta}H^{m-j}\psi\|_{2+|\alpha|,0}^{\lambda(\alpha)}\|\partial^{\beta}H^{m-j}\psi\|^{1-\lambda(\alpha)} \\ &\leq C\|H^{m-j}\psi\|_{2+|\alpha+\beta|,0}^{\lambda(\alpha)}\|H^{m-j}\psi\|_{|\beta|,0}^{1-\lambda(\alpha)}\end{aligned}$$

so that

$$\|H^{j-1}\tilde{\nabla}H^{m-j}\psi\| \le C \sum_{|\alpha|\le 2j-2} \|H^{m-j}\psi\|_{2j,0}^{\lambda(\alpha)}\|H^{m-j}\psi\|_{2j-2,0}^{1-\lambda(\alpha)}$$

$$\le C\|H^{m-j}\psi\|_{2j,0}^{\mu_j}\|H^{m-j}\psi\|_{2j-2,0}^{1-\mu_j}, \qquad (3.9)$$

where $\mu_j = \max_{|\alpha|\le 2j-2} \lambda(\alpha)$ Again by the Heinz-Kato theorem and the moment inequality [26], for any $\ell \le 2j$,

$$\|H^{m-j}\psi\|_{\ell,0} \le C\||H|^{m-j+\ell/2}\psi\| + C\|H^{m-j}\psi\|$$

$$\le C\|H^m\psi\|^{(m-j+\ell/2)/m}\|\psi\|^{(j-\ell/2)/m} + C\|H^m\psi\|^{1-j/m}\|\psi\|^{j/m}$$

and hence, the R.H.S. of the last inequality in (3.9) is bounded by $C(\|H^m\psi\|+\|\psi\|)^{1-(1-\mu_j)/m}\|\psi\|^{(1-\mu_j)/m}$ This proves part (1)

(2) We first consider the case $j = 0$. There is a constant $0 \le \lambda_0 < 1$ such that

$$\|\tilde{\nabla}H^m\psi\|_{-1,0} \le C\|H^m\psi\|_{1,0}^{\lambda_0}\|H^m\psi\|^{1-\lambda_0}$$

By Lemma 3.5 and (3.2),

$$\|H^m\psi\|_{1,0} \le C\|\nabla H^m\psi\| + C\|H^m\psi\| \le C\|\nabla H^m\psi\| + C\|\psi\|,$$

from which we have the required inequality We proceed to the case $1 \le j \le m$. Noting that $H \in \mathcal{L}(H^{1,0}; H^{-1,0})$ and $D(|H|^{\ell/2}) = H^{\ell,0}$ for any $1 \le \ell \le 2m+1$, we obtain

$$\|H^j\tilde{\nabla}H^{m-j}\psi\|_{-1,0} \le C\|H^{j-1}\tilde{\nabla}H^{m-j}\psi\|_{1,0}$$

$$\le C\||H|^{j-1/2}\tilde{\nabla}H^{m-j}\psi\| + C\|H^{j-1}\tilde{\nabla}H^{m-j}\psi\|$$

$$\le C \sum_{|\alpha+\beta|\le 2j-1} \|\partial^\alpha\tilde{\nabla}\cdot\partial^\beta H^{m-j}\psi\|$$

In the same way as in part (1), the R.H.S. of the last inequality is estimated by $C\|\psi\|_{2m+1,0}^{\lambda_j}\|\psi\|^{1-\lambda_j}$, where $0 \le \lambda_j < 1$. The required inequality then follows from

$$\|\psi\|_{2m+1,0} \le C\|\nabla H^m\psi\| + C\|\psi\|,$$

which is an consequence of Lemma 3.5 and (3.2) Q.E.D.

The following lemma in an essential part of this section. The argument in the proof will clarify the reason why we have imposed the condition $\lambda < 1$ in $\Sigma_k$.

Lemma 3.11. Let $k \in \mathbb{N}$ and let $\phi \in \mathcal{H}_k$. Then,

$$\||J|^k(t)e^{-itH}\phi\| \le C(1+|t|)^k\|\phi\|_{0,k}, \quad t \in \mathbb{R}.$$

Proof. We consider the nontrivial case $t \neq 0$. For simplicity we assume $t > 0$, since the case $t < 0$ can be treated analogously. We distinguish between the cases $k$ = even and $k$ = odd. We first prove the lemma for $k = 2m$, $m \in \mathbb{N}$. By Lemma 3.5,

$$\begin{aligned}\||J|^k(t)e^{-itH}\phi\| &= t^{2m}\|(-\Delta)^m S(-t)e^{-itH}\phi\| \\ &\le C(2t^2)^m(\|H^m S(-t)e^{-itH}\phi\| + \|S(-t)e^{-itH}\phi\|) \\ &= C(\|K^m(t)e^{-itH}\phi\| + (2t^2)^m\|\phi\|)\end{aligned}$$

Therefore it suffices to prove

$$\|K^m(t)e^{-itH}\phi\| \le C(1+t)^{2m}\|\phi\|_{0,2m}, \quad t > 0. \tag{3.10}$$

By Lemma 3.8,

$$\begin{aligned}
\frac{d}{dt}\|\chi^m(t)\phi\|^2 &= 2\mathrm{Re}\left(\frac{d}{dt}\chi^m(t)\phi, \chi^m(t)\phi\right) \\
&= 8t\mathrm{Re}\sum_{j=1}^{m}(\chi^{j-1}(t)\tilde{\gamma}(t)\chi^{m-j}(t)\phi, \chi^m(t)\phi) \\
&= 8t\mathrm{Re}\sum_{j=1}^{m}(K^{j-1}(t)\tilde{V}K^{m-j}(t)e^{-itH}\phi, K^m(t)e^{-itH}\phi) \\
&= 2^{2m+2}t^{4m-1}\mathrm{Re}\sum_{j=1}^{m}(H^{j-1}\tilde{V}H^{m-j}S(-t)e^{-itH}\phi, H^m S(-t)e^{-itH}\phi)
\end{aligned}$$

By Lemma 3.10, we obtain for some $0 \le \lambda < 1$,

$$\begin{aligned}
\left|\frac{d}{dt}\|\chi^m(t)\phi\|^2\right| &\le Ct^{4m-1}(\|H^m S(-t)e^{-itH}\phi\| + \|\phi\|)^{\lambda}\|\phi\|^{1-\lambda}\|H^m S(-t)e^{-itH}\phi\| \\
&\le Ct^{1-2\lambda}(\|\chi^m(t)\phi\| + (2t^2)^m\|\phi\|)^{\lambda}(t^{2m-2}\|\phi\|)^{1-\lambda}\|\chi^m(t)\phi\|
\end{aligned}$$

and therefore

$$\left|\frac{d}{dt}\|\chi^m(t)\phi\|\right| \le Ct^{1-2\lambda}(t^{2m-2}\|\phi\|)^{1-\lambda}\|\chi^m(t)\lambda\|^{\lambda} + Ct^{2m-1}\|\phi\| \tag{3.11}$$

The R.H.S. of (3.11) is estimated by

$$Ct^{1-2\lambda}(t^{2m-2}\|\phi\| + \|\chi^m(t)\phi\|) + Ct^{2m-1}\|\phi\|,$$

so that by integration over an interval $[0,t]$,

$$\|\chi^m(t)\phi\| \le \||x|^{2m}\phi\| + C(1+t)^{2m}\|\phi\| + C\int_0^t s^{1-2\lambda}\|\chi^m(s)\phi\|\,ds.$$

By Gronwall's lemma,

$$\|\mathcal{X}^m(t)\phi\| \le C\exp(C(1+t))(\||x|^{2m}\phi\| + (1+t)^{2m}\|\phi\|) \qquad (3.12)$$

This shows (3.10) for $t \le 1$. Now let $t \ge 1$. By (3.11),

$$\frac{d}{dt}(t^{-2m+1}\|\mathcal{X}^m(t)\phi\|)$$

$$\le -(2m-1)t^{-2m}\|\mathcal{X}^m(t)\phi\| + C\|\phi\|^{1-\lambda}(t^{-2m}\|\mathcal{X}^m(t)\phi\|)^{\lambda} + C\|\phi\| \le C\|\phi\|$$

By integration over an interval $[1,t]$, we obtain from (3.12)

$$t^{1-2m}\|\mathcal{X}^m(t)\phi\| \le \|\mathcal{X}^m(1)\phi\| + Ct\|\phi\| \le C(1+t)\|\phi\|, \ t \ge 1.$$

This shows (3.10) for $t \ge 1$, as desired.

We next consider the case $k = 2m+1$, $m \in \mathbb{N}\cup\{0\}$. By Lemma 3.5,

$$\begin{aligned}\||J|^k(t)e^{-itH}\phi\|^2 &= t^{2k}\|(-\Delta)^{k/2}S(-t)e^{-itH}\phi\|^2 \\ &\le Ct^{2k}\Big((1/2)\|\nabla H^m S(-t)e^{-itH}\phi\|^2 \\ &\quad + (VH^m S(-t)e^{-itH}\phi, H^m S(-t)e^{-itH}\phi)\Big) + Ct^{2k}\|\phi\|^2 \\ &\le Ct^{2k}\|\nabla H^m S(-t)e^{-itH}\phi\|^2 + Ct^{2k}\|\phi\|^2 \\ &\le C\|J(t)K^m(t)e^{-itH}\phi\|^2 + Ct^{2k}\|\phi\|^2\end{aligned}$$

Therefore it suffices to prove

$$\|J(t)K^m(t)e^{-itH}\phi\|^2 \le C(1+t)^{2k}\|\phi\|^2_{0,k}, \ t > 0. \qquad (3.13)$$

We note here that (3.2) implies

$$C^{-1}(\|J(t)K^m(t)e^{-itH}\phi\|^2 + 2^{4m}t^{2k}\|\phi\|^2)$$

$$\le (1/2)\|J(t)K^m(t)e^{-itH}\phi\|^2 + t^2(VK^m(t)e^{-itH}\phi, K^m(t)e^{-itH}\phi) + 2^{4m}t^{2k}\|\phi\|^2$$

$$\le C(\|J(t)K^m(t)e^{-itH}\phi\|^2 + 2^{4m}t^{2k}\|\phi\|^2) \tag{3.14}$$

Moreover,

$$\|J(t)K^m(t)e^{-itH}\phi\|^2 + 2t^2(VK^m(t)e^{-itH}\phi, K^m(t)e^{-itH}\phi)$$

$$= (\mathcal{K}^{m+1}(t)\phi, \mathcal{K}^m(t)\phi), \tag{3.15}$$

where ( , ) denotes the pairing between $\mathcal{H}_1'$ and $\mathcal{H}_1$ By Lemmas 3.8 and 3.9,

$$\frac{d}{dt}(\mathcal{K}^{m+1}(t)\phi, \mathcal{K}^m(t)\phi)$$

$$= 4t(\hat{\mathcal{V}}(t)\mathcal{K}^m(t)\phi, \mathcal{K}^m(t)\phi) + 8t\mathrm{Re}\sum_{j=1}^{m}(\mathcal{K}^j(t)\hat{\mathcal{V}}(t)\mathcal{K}^{m-j}(t)\phi, \mathcal{K}^m(t)\phi)$$

$$= 2^{2m+2}t^{4m+1}(\tilde{V}H^mS(-t)e^{-itH}\phi, H^mS(-t)e^{-itH}\phi)$$

$$+ 2^{2m+3}t^{4m+1}\mathrm{Re}\sum_{j=1}^{m}(H^j\tilde{V}H^{m-j}S(-t)e^{-itH}\phi, H^mS(-t)e^{-itH}\phi)$$

By Lemma 3.10, we obtain for some $0 \le \lambda < 1$,

$$\left|\frac{d}{dt}(\mathcal{K}^{m+1}(t)\phi, \mathcal{K}^m(t)\phi)\right|$$

$$\le Ct^{4m+1}(\|\nabla H^mS(-t)e^{-itH}\phi\| + \|\phi\|)^{1+\lambda}\|\phi\|^{1-\lambda}$$

$$\le Ct^{-\lambda}(t^{2m}\|\phi\|)^{1-\lambda}\|J(t)K^m(t)e^{-itH}\phi\|^{1+\lambda} + Ct^{4m+1}\|\phi\|^2 \tag{3.16}$$

The R.H.S. of (3.16) is estimated by

$$Ct^{-\lambda}(t^{4m}\|\phi\|^2 + \|J(t)K^m(t)e^{-itH}\phi\|^2) + Ct^{4m+1}\|\phi\|^2,$$

so that by integration over an interval $[0,t]$ and (3.14)-(3.15),

$$\|J(t)K^m(t)e^{-itH}\phi\|^2 \le C(\mathcal{X}^{m+1}(t)\phi, \mathcal{X}^m(t)\phi) + Ct^{2k}\|\phi\|^2$$

$$\le C\||x|^k\phi\|^2 + C(1+t)^{2k}\|\phi\|^2 + C\int_0^t s^{-\lambda}\|J(s)K^m(s)e^{-isH}\phi\|^2\, ds.$$

By Gronwall's lemma,

$$\|J(t)K^m(t)e^{-itH}\phi\|^2 \le C\exp(C(1+t))(\||x|^k\phi\|^2 + (1+t)^{2k}\|\phi\|^2) \tag{3.17}$$

Let $t \ge 1$. By (3.14)-(3.16), we have for $\varepsilon > 0$ sufficiently small

$$\frac{d}{dt}(t^{-4m-1}(\mathcal{X}^{m+1}(t)\phi, \mathcal{X}^m(t)\phi))$$

$$\le -(4m+1)t^{-4m-2}(\mathcal{X}^{m+1}(t)\phi, \mathcal{X}^m(t)\phi)$$

$$+ C(t^{-4m-2}\|J(t)K^m(t)e^{-itH}\phi\|^2)^{(1+\lambda)/2}\|\phi\|^{1-\lambda} + C\|\phi\|^2$$

$$\le - Ct^{-4m-2}\|J(t)K^m(t)e^{-itH}\phi\|^2 + C\|\phi\|^2$$

$$+ \varepsilon t^{-4m-2}\|J(t)K^m(t)e^{-itH}\phi\|^2 + C(\varepsilon)\|\phi\|^2 \le C(\varepsilon)\|\phi\|^2$$

Integration over an interval $[1,t]$, (3.14), (3.15) and (3.17) yield (3.13) for $t \ge 1$. Q.E.D.

Proof of Theorems 1-2. Let $k \in \mathbb{N}$ and let $\phi \in H^{0,k}$ There is a sequence $\{\phi_j\}$ in $\mathcal{K}_k$ such that $\phi_j \longrightarrow \phi$ in $H^{0,k}$ as $j \longrightarrow \infty$. By Lemma 3.4, $\mathbb{R} \ni t \longmapsto |J|^k(t)e^{-itH}\phi_j \in L^2$ is continuous. It follows from Lemma 3.11 that

$$\||x|^kU(-t)e^{-itH}\phi_j\| = \||J|^k(t)e^{-itH}\phi_j\|$$

$$\le C(1+|t|)^k\|\phi_j\|_{0,k}, \tag{3.18}$$

$$\| |x|^k U(-t)e^{-itH}(\phi_j - \phi_\ell)\| = \| |J|^k(t)e^{-itH}(\phi_j - \phi_\ell)\|$$

$$\le C(1+|t|)^k \|\phi_j - \phi_\ell\|_{0,k}. \tag{3.19}$$

On the other hand,

$$\sup_{t\in\mathbb{R}} \|U(-t)e^{-itH}(\phi_j - \phi)\| = \|\phi_j - \phi\| \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.$$

Since the multiplication operator by $|x|^k$ is closed in $L^2$, we see that $U(-t)e^{-itH}\phi \in H^{0,k}$ and $|x|^k U(-t)e^{-itH}\phi_j \longrightarrow |x|^k U(-t)e^{-itH}\phi$ in $L^2$ as $j \longrightarrow \infty$ Therefore by (3.18)-(3.19) we obtain

$$\| |J|^k(t)e^{-itH}\phi\| \le C(1+|t|)^k \|\phi\|_{0,k}, \tag{3.20}$$

$$\| |J|^k(t)e^{-itH}(\phi_j - \phi)\| \le C(1+|t|)^k \|\phi_j - \phi\|_{0,k}.$$

This proves that $\mathbb{R} \ni t \longmapsto |J|^k(t)e^{-itH}\phi \in L^2$ is continuous.

For $|\alpha| \le k$, we set $\mu = |\alpha|/k$. By Hölder's inequality and (3.20),

$$\|J^\alpha(t)e^{-itH}\phi\| = \|x^\alpha U(-t)e^{-itH}\phi\|$$

$$\le \| |x|^k U(-t)e^{-itH}\phi\|^\mu \|U(-t)e^{-itH}\phi\|^{1-\mu} \le C(1+|t|)^{|\alpha|} \|\phi\|_{0,k}$$

Similarly, we have for $t, s \in \mathbb{R}$

$$\|J^\alpha(t)e^{-itH}\phi - J^\alpha(s)e^{-isH}\phi\|$$

$$\le \|(U(t)-U(s))x^\alpha U(-s)e^{-isH}\phi\| + \|x^\alpha(U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi)\|,$$

$$\|x^\alpha(U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi)\|$$

$$\le \| |x|^k(U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi)\|^\mu \|U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi\|^{1-\mu},$$

$$\| |x|^k (U(-t)e^{-itH}\phi - U(-s)e^{-isH}\phi) \|$$

$$\le \| |J|^k(t)e^{-itH}\phi - |J|^k(s)e^{-isH}\phi \| + \| (U(s)-U(t)) |x|^k U(-s)e^{-isH}\phi \|$$

Collecting these estimates, we obtain

$$\| J^\alpha(t)e^{-itH}\phi - J^\alpha(s)e^{-isH}\phi \| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow s.$$

We have thus proved $(J)_k$. Proposition 2.1 then completes the proof of Theorems 1-2. Q.E.D.

Proof of Theorem 3. It remains to prove part (4) Let $\phi \in H^{0,\infty}$ It suffices to prove that for any $k \in \mathbb{N}$ and $t \neq 0$, $H^k e^{-itH}\phi \in C^\infty$ and $\mathbb{R}\setminus\{0\} \ni t \longmapsto H^k e^{-itH}\phi \in C^\infty$ is continuous. We have

$$H^k e^{-itH}\phi = \sum_{j=0}^{k} \binom{k}{j} V^j H_0^{k-j} e^{-itH}\phi$$
$$+ \sum_{j=1}^{k-1} \sum_{\substack{|\beta| \le 2(k-j)-1 \\ |\alpha_1 + \cdots + \alpha_j + \beta| = 2(k-j)}} C(k,j,\{\alpha_\ell\},\beta) \left( \prod_{\ell=1}^{j} \partial^{\alpha_j} V \right) \partial^\beta e^{-itH}\phi \qquad (3.21)$$

and therefore Proposition 2.2 implies that $H^k e^{-itH}\phi \in C^\infty$, $t \neq 0$ We next prove the required continuity Let $t, s \neq 0$. With the notation as in (2.4), we have by (3.21)

$$|H^k e^{-itH}\phi - H^k e^{-isH}\phi|_{m,R} \le C|e^{-itH}\phi - e^{-isH}\phi|_{m+2k,R}.$$

The last semi-norm is estimated as in (2.5) with $m$ and $\psi$ replaced by $m + 2k$ and $\phi$, respectively This proves the continuity at $s$.

Q.E.D.

## Appendix. Some inequalities for $L^p_{unif}$-potentials.

In this appendix we prove the properties of $L^p_{unif}$-potentials mentioned in Example 2. See, CYCON, FROESE, KIRSCH & SIMON [3; § 1.2], SIMADER [22; Hilfssatz 1], SIMON [23; §25], STRICHARTZ [25; Chapter II] for some other results on $L^p_{unif}$ spaces.

Proposition A.1. Let $m \geq 2$ be an integer. Let $p \in [2,\infty)$ satisfy $p > n/m$. Let $V \in L^p_{unif}$. Then for any $\psi \in H^{m,0}$

$$\|V\psi\| \leq C\|V\|_{L^p_{unif}} \|\psi\|_{m,0}^{n/mp}\|\psi\|^{1-n/mp}$$

Proof. We prove the proposition by making a modification of an argument of BREZIS & KATO [1; Remark 2.1]. We denote by $B(x)$ the unit ball in $\mathbb{R}^n$ with center at $x$ and by $\chi_{B(x)}$ the characteristic function of $B(x)$. Let $\xi \in C_0^\infty$ satisfy $\xi \geq 0$, $\operatorname{supp} \xi \subset B(0)$ and $\|\xi\| = 1$. We define the translation $\tau_x\xi$ by $\tau_x\xi(y) = \xi(y-x)$. Let $q \in (2,\infty]$ satisfy $1/q = 1/2 - 1/p$.
We have by Hölder's inequality

$$\begin{aligned}
\|V\psi\|^2 &= \int \left(\int |V(y)|^2|\xi(y-x)\psi(y)|^2\, dx\right) dy \\
&= \int \left(\int |V(y)\chi_{B(x)}(y)|^2|(\tau_x\xi)(y)\psi(y)|^2\, dy\right) dx \\
&\leq \int \|V\chi_{B(x)}\|^2_{L^p} \|\tau_x\xi\cdot\psi\|^2_{L^q}\, dx \\
&\leq \|V\|^2_{L^p_{unif}} \int \|\tau_x\xi\cdot\psi\|^2_{L^q}\, dx. \qquad \text{(A.1)}
\end{aligned}$$

By the Gagliardo-Nirenberg inequality [4], for $a = n/mp$,

$$\|\tau_x \xi \cdot \psi\|_{L^q}^2 \le C \left( \sum_{|\alpha|=m} \|\partial^\alpha (\tau_x \xi \cdot \psi)\|^2 \right)^a \|\tau_x \xi \cdot \psi\|^{2(1-a)} \tag{A.2}$$

We have

$$\int \sum_{|\alpha|=m} \|\partial^\alpha (\tau_x \xi \cdot \psi)\|^2 \, dx$$

$$\le C \sum_{|\beta+\gamma|=m} \int \left( \int |(\partial^\beta \xi)(y-x)|^2 |\partial^\gamma \psi(y)|^2 \, dy \right) dx$$

$$= C \sum_{|\beta+\gamma|=m} \|\partial^\beta \xi\|^2 \|\partial^\gamma \psi\|^2 \le C \|\psi\|_{m,0}^2, \tag{A.3}$$

$$\int \|\tau_x \xi \cdot \psi\|^2 \, dx = \|\psi\|^2 \tag{A.4}$$

Now let $\mu > 0$ By (A.2)-(A.4),

$$\int \|\tau_x \xi \cdot \psi\|_{L^q}^2 \, dx$$

$$\le C \int \left(\mu \sum_{|\alpha|=m} \|\partial^\alpha (\tau_x \xi \cdot \psi)\|^2 + \mu^{-a/(1-a)} \|\tau_x \xi \cdot \psi\|^2\right) dx$$

$$\le C \left(\mu \|\psi\|_{m,0}^2 + \mu^{-a/(1-a)} \|\psi\|^2\right)$$

Minimization of the R.H.S. of the last inequality with respect to $\mu$ gives

$$\int \|\tau_x \xi \cdot \psi\|_{L^q}^2 \, dx \le C \|\psi\|_{m,0}^{2a} \|\psi\|^{2(1-a)} \tag{A.5}$$

Estimates (A.1) and (A.5) prove the proposition. Q.E.D.

<u>Proposition A.2.</u> Let $p \in [1,\infty)$ satisfy $p > n/2$. Let $V \in L^p_{unif}$. Then for any $\psi_1, \psi_2 \in H^{1,0}$

$$|(V\psi_1,\psi_2)| \le C\|V\|_{L^p_{unif}} \prod_{j=1}^{2} \|\psi_j\|_{1,0}^{n/2p} \|\psi_j\|^{1-n/2p}$$

In particular, for any $\psi \in H^{1,0}$,

$$\|V\psi\|_{-1,0} \le C\|V\|_{L^p_{unif}} \|\psi\|_{1,0}^{n/2p} \|\psi\|^{1-n/2p}$$

<u>Proof.</u> Since $|(V\psi_1,\psi_2)| \le \prod_{j=1}^{2} |(|V|\psi_j,\psi_j)|^{1/2}$, it suffices to prove

$$\int |V(y)||\psi(y)|^2\, dy \le C\|V\|_{L^p_{unif}} \|\psi\|_{1,0}^{n/p} \|\psi\|^{2-n/p}, \quad \psi \in H^{1,0},$$

which can be verified in the same way as in the proof of Proposition A.1. Q.E.D.

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